Lecture 23 (next Monday) will be last lecture

next Wed, there will be final review (during lecture)

this Fri, HW13 is due, then Sat, HW14 will drop
← last HW of semester

post Exam 2 review material will be posted by EOD saturday
(meanwhile, use existing review packets)

Final Exam will be cumulative with an emphasis on post-Exam 2 topics.

Determinants cont'd:

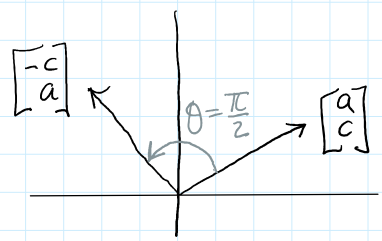
thm: 2x2 matrix $A = [\vec{v}\ \vec{w}]$ where $\vec{v}, \vec{w}$ are non-zero column vectors

$\det[\vec{v}\ \vec{w}] = \|\vec{v}\| \sin\theta \|\vec{w}\|$ or $\|\vec{v}\|\|\vec{w}\|\sin\theta$ why?

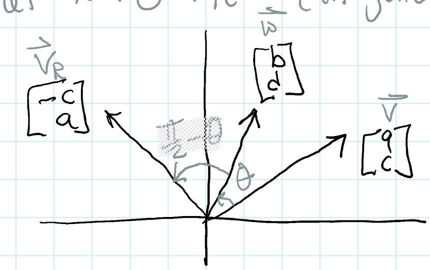
let $\vec{v} = \begin{bmatrix} a \\ c \end{bmatrix}$, $\vec{w} = \begin{bmatrix} b \\ d \end{bmatrix} \Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\downarrow$ 90° CCW rotation

$\vec{v}_R \begin{bmatrix} -c \\ a \end{bmatrix}$



let $-\pi < \theta < \pi$ (in general)



$$\det A = ad - bc$$
$$= \vec{v}_R \cdot \vec{w}$$
$$= \|\vec{v}_R\|\|\vec{w}\|\cos\left(\tfrac{\pi}{2} - \theta\right)$$

$\vec{v}_R \cdot \vec{w}$: $\begin{bmatrix} -c \\ a \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = -cb + ad = ad - bc$

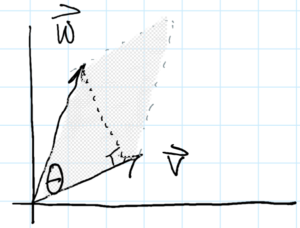
recall: $\cos\theta = \dfrac{\vec{v}_R \cdot \vec{w}}{\|\vec{v}_R\|\|\vec{w}\|}$ then $\|\vec{v}_R\|\|\vec{w}\|\cos\theta = \vec{v}_R \cdot \vec{w}$
$\uparrow$ $\frac{\pi}{2} - \theta$

$\downarrow$ same length $\qquad \downarrow$ complementary ∡'s

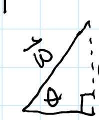
$$\det A = \|\vec{v}\|\|\vec{w}\| \sin\theta$$

Revelation:

$|\det A|$ = area of parallelogram spanned by $\vec{v}, \vec{w}$



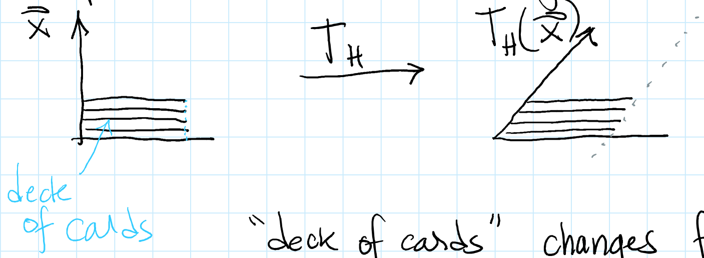
$\text{AREA}_{\square} = bh$
$b = \|\vec{v}\|$
$h = \|\vec{w}\| \sin\theta$



$\sin\theta = \dfrac{\text{OPP}}{\|\vec{w}\|}$

$\|\vec{w}\| \sin\theta = \text{OPP} = h$

explore the linear transformation: shear explore the linear transformation : shear

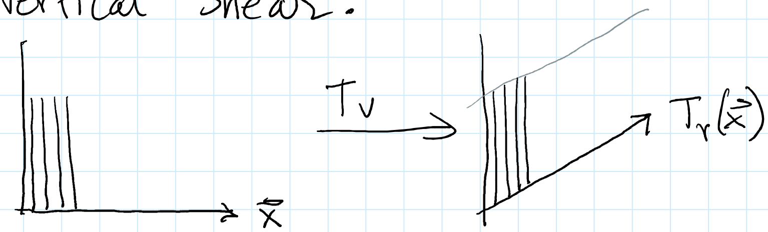


horizontal shear:

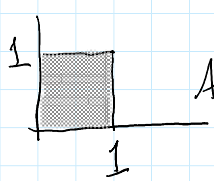
$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ where $k :=$ arbitrary constant

"deck of cards" changes from rectangle to a parallelogram when T_H is applied

Vertical Shear:



$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$



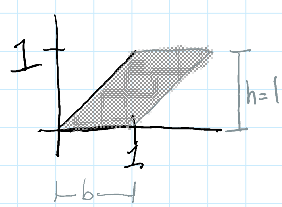
Area = 1

$[\vec{e}_1, \vec{e}_2]$

let $k = 1$:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

still $\vec{e}_1$



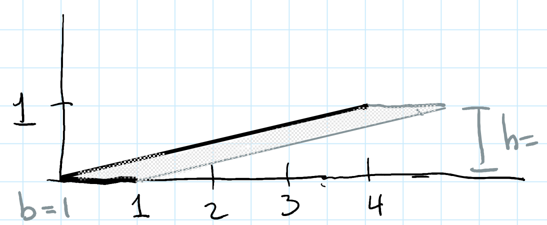
Area

$A = bh$
$= 1 \cdot 1 \quad = 1$

let $k = 4$:

$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

A

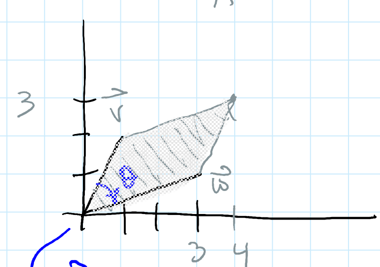


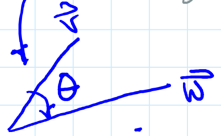
Area = 1

also $\det A = 1$

ex. given $A = \begin{array}{cc} \vec{v} & \vec{w} \\ \end{array} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$\vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$





negative rotation

area$_{▱}$ $= |\det A|$
$= |1 - 6|$ ← -5
$= 5$

$\det A > 0$ if $0 < \theta < \pi$

$\det A < 0$ if $-\pi < \theta < 0$

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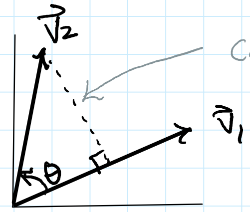
$\det A = 0$ if $\vec{v} \parallel \vec{w} \Rightarrow \theta = 0$ or $\theta = \pi$



can be written as $\vec{v}_{2\perp}$

then, by applying Gram-Schmidt process,

$$|\det A| = \|\vec{v}_1\| \, \underline{\|\vec{v}_2\| \, |\sin\theta|}$$
$$= \|\vec{v}_1\| \, \|\vec{v}_{2\perp}\|$$

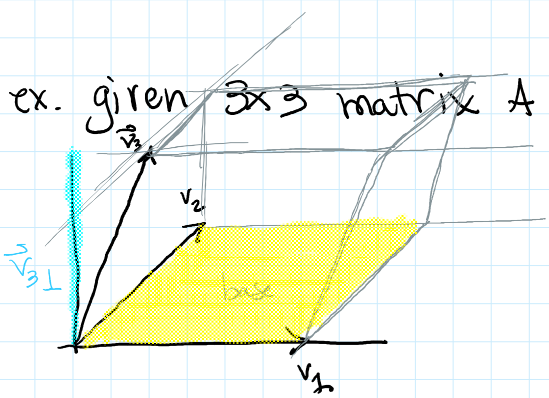
Expand Definition* of Determinant

given $n \times n$ matrix A with columns $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

$$|\det A| = \|\vec{v}_1\| \, \|\vec{v}_{2\perp}\| \dots \|\vec{v}_{n\perp}\|$$

where $\vec{v}_{i\perp}$ is component of $\vec{v}_i$ and perpendicular to $\text{span}(\vec{v}_1, \dots, \vec{v}_{i-1})$

ex. given 3×3 matrix $A = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3]$



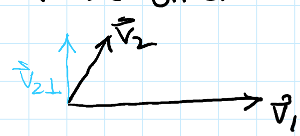
parallelepiped defined $\vec{v}_1, \vec{v}_2, \vec{v}_3$

volume $= [\text{area base}](\text{height})$

$= [\text{area base}] \; \|\vec{v}_{3\perp}\|$

$= \underline{\|\vec{v}_1\| \, \|\vec{v}_{2\perp}\|} \; \|\vec{v}_{3\perp}\|$

recall sheer:



area: $\|\vec{v}_1\| \, \|\vec{v}_{2\perp}\|$

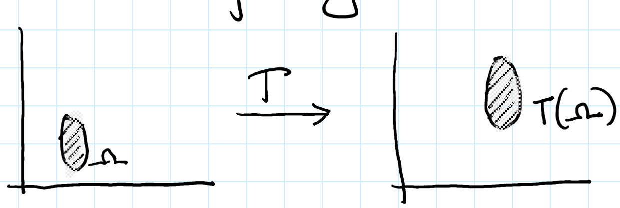
conclusion: above def'n* holds ☺

Expansion Factor

shows what determinant tell us about the area of a region after LT is applied

given transformation T ⟶ can be an combination of:
- scalar
- reflection
- rotation
- sheer
- projection
- ⋮

area of region Ω in $\mathbb{R}^2$



Q: how is the area of Ω impacted by T?

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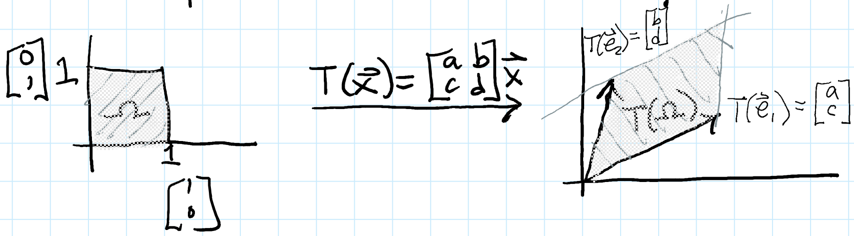
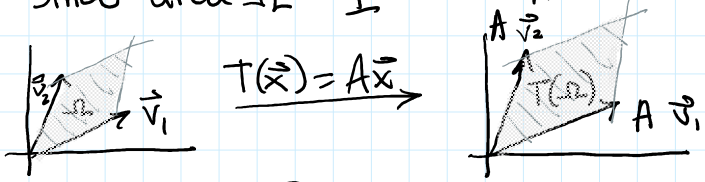
explore the ratio $\dfrac{\text{area of } T(\Omega)}{\text{area of } \Omega}$



since area $\Omega = 1$ know $\text{area}_{T(\Omega)} = \text{area}_{▱} = |\det A|$



Let $B = [\vec{v}_1\ \vec{v}_2]$

$\therefore \text{area}_{\Omega} = |\det B|$

$$\begin{aligned}
\text{area}_{T(\Omega)} &= |\det[A\vec{v}_1\ A\vec{v}_2]| \\
&= |\det[A[\vec{v}_1, \vec{v}_2]]| \\
&= |\det[AB]| \\
&= |\det A \det B| \\
&= |\det A|\,|\det B|
\end{aligned}$$

$$\therefore \frac{\text{area}_{T(\Omega)}}{\text{area}_{\Omega}} = \frac{|\det A|\,\cancel{|\det B|}}{\cancel{|\det B|}} = |\det A|$$

Expansion Factor given $T(\vec{x}) = A\vec{x} : \mathbb{R}^2 \to \mathbb{R}^2$

then $|\det A|$ is the expansion factor $\dfrac{\text{area}_{T(\Omega)}}{\text{area}_{\Omega}}$ of T

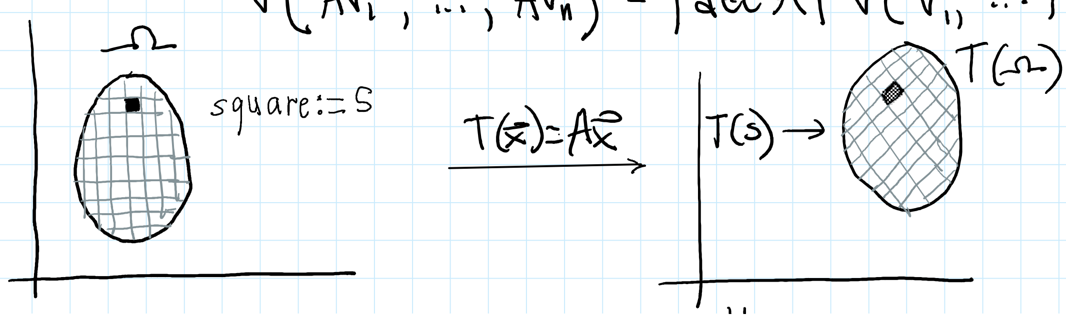
on the set of parallelograms Ω

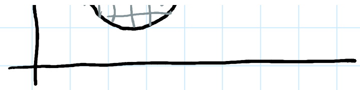
Likewise, for $T(\vec{x}) = A\vec{x} : \mathbb{R}^n \to \mathbb{R}^n$

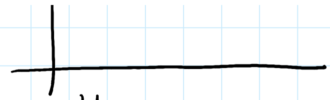
then $|\det A|$ is expansion factor of T

on the set of parallelepipeds:

$$V(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A|\, V(\vec{v}_1, \dots, \vec{v}_n)$$







then $\dfrac{\text{area}_{T(s)}}{\text{area } S}$

$= |\det A|$

ratio indicates how square unit, S, got scaled